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Analytic solutions of the radial pulsation equation for rotating and magnetic star models

Richard Stothers *NASA Institute for Space Studies, Goddard Space Flight Center, 2880 Broadway, New York, NY 10025, USA*

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Summary. The wave equation describing small radial perturbations of spherically symmetric, gaseous stars has been generalized to include, in a simple way, the effects of axial rotation and of tangled magnetic fields. Solutions in closed form have been obtained for the adiabatic pulsation periods of five analytic stellar models in two special cases, which have, none the less, considerable astrophysical interest. Non-adiabatic stability criteria have been determined by means of the one-zone stellar model. Results are discussed for a range of physical parameters of the models, and applications are made to the case of classical Cepheids and other variable giant stars.

1 Introduction

The wave equation that describes small radial oscillations of self-gravitating gas spheres admits solutions in closed form in the case of a few specialized stellar models. Although these models are very simplified approximations to real astronomical bodies, the solutions do provide a useful and illuminating means of interpreting the much more complex numerical results obtained for more realistic cases, and have some mathematical interest in their own right. The five models for which adiabatic pulsational periods have been derived completely analytically can most conveniently be described in terms of their interior density distributions: (1) the homogeneous model, with $\rho = \text{constant}$; (2) the inverse-square model, with $\rho = \rho_R (r/R)^{-2}$; (3) the Roche model, containing a point mass surrounded by an envelope of negligible mass in which $\rho = \rho_R (r/R)^{-2}$; (4) the Prasad model, consisting of a point mass surrounded by an envelope which contains two-thirds of the total mass and has $\rho = \text{constant}$; and (5) an atmospheric model, which is similar to the Roche model except that its density distribution is determined from the requirement that the temperature lapse rate be equal to a constant (variants of this model exist).

The first three solutions in their most general form were originally obtained by Sterne (1937) (see also Ritter 1879; Miller 1929; Kopal 1948; Rosseland 1949; Vaughan 1972). The fourth and fifth solutions were derived by Prasad (1948) and by Lamb (1932), respectively, although Lamb's solution has precedents that he acknowledges. A more useful form of Lamb's solution appeared in a paper by Gough, Ostriker & Stobie (1965). No additional solutions in closed form are known. Van der Borght (1970), however, did try general series expansions that gave rise to two-term recurrence relations; Murphy & Smith (1970) tried, also unsuccessfully, three-term recurrence relations.

Pulsational and secular stability has been studied analytically in the case of the homogeneous model, usually in the 'one-zone' approximation (Jeans 1927, 1929; Baker 1966). In both Jeans's and Baker's investigations, the destabilization provided by variations of the opacity and of the gas and radiation thermodynamics has been demonstrated very clearly. Subatomic energy release has also been considered by Jeans, while turbulent convection has been investigated by other authors (Cowling 1935; Unno & Kamijo 1966; Unno 1967; Gough 1967). It fortunately turns out that the neglect of stratification in the stellar models has no significant effect on the basic criteria for the overall stability of the star (Okamoto & Unno 1967; see also Ledoux 1963).

In previous investigations, axial rotation and magnetic fields were ignored. Both forces are of course non-radial in general, and are therefore difficult to handle analytically. In the rotational case, however, if the angular velocity of rotation is much less than the fundamental pulsational eigenfrequency, the star can, to a first approximation, be considered as spherically symmetric and subject to mostly radial oscillations (Ledoux 1945). Furthermore, the change of structure of the star due to rotation can be estimated by the 'mean sphere' approximation, even though the change of shape of the star is ignored (Monaghan 1968). This approximation is applicable to the case of either uniform or differential rotation. In giant variable stars like classical Cepheids, rotation is undoubtedly rather slow throughout the pulsating layers (Kraft 1966), and therefore the oscillations can be treated as being pseudo-radial. In the magnetic case, the spherically symmetric approximation holds only (1) if the total magnetic energy of the star is much less than its gravitational potential energy (Chandrasekhar & Fermi 1953) or (2) if the magnetic field is either well tangled or otherwise distributed axisymmetrically so that $\langle H_r^2 \rangle = (1/3)\ \langle H^2 \rangle$ (where angular brackets denote an average taken over a spherical shell) (Trasco 1970). A tangled magnetic field is not an unreasonable assumption for the outer, pulsating layers of giant stars like classical Cepheids. Such a magnetic field could have arisen from dynamo action in the convective envelope during the red-giant phase that immediately preceded the Cepheid phase, or else in the central convective core during the still earlier main-sequence phase or even during the present Cepheid phase itself, the magnetic flux tubes being continually buoyed up to the surface (Parker 1955; Jensen 1955). Alternatively, a primordial magnetic field could have been built up, intensified by flux conservation, and twisted by convection and rotation during the pre-main-sequence collapse phase. Certainly it now appears that a fairly complex field geometry is needed for the stabilization of even a very strong magnetic field within a star (Tayler 1974). Although the magnetic field in the main pulsating layers of a Cepheid is likely to be small-scale, weak, and dissipative, some field may always be present because of continual replenishment of the magnetic flux from deeper layers. At least the observed magnetic fields in a few Cepheids are significant enough to be astrophysically interesting (Borra 1981). Thus, rotation and magnetic fields may turn out to contribute, in part, to the solution of the well-known problem of the Cepheid mass discrepancy (Carson & Stothers 1976; Stothers 1979b).

The present paper addresses the question of how the adiabatic period spectrum and the non-adiabatic stability criteria for analytic models of variable stars undergoing small, spherically symmetric oscillations are affected by the inclusion of axial rotation and magnetic fields in the approximations discussed above. Although these approximations are mathematically necessary in order to obtain completely analytic solutions, they are not excessively restrictive in many applications of astrophysical interest, such as the case of the Cepheid variables, and some insight into the behaviour of real stars is expected to be obtained. It must be pointed out, however, that any assumptions about the rotational and magnetic fields that are radically different from those made here may lead to very different results. In the particular case of the stellar atmosphere model, the present assumptions must always break down, and results for this model can have only rough heuristic value.

2 Periods

2.1 THE PREVIOUSLY KNOWN ANALYTIC SOLUTIONS

Eddington's (1918, 1926) form of the radial adiabatic wave equation, neglecting rotation and magnetic fields, can be written

$$\frac{d^2\eta}{dr^2} + \left(\frac{4-V}{r}\right)\frac{d\eta}{dr} + \left(\frac{\sigma^2\rho}{\gamma P} - \frac{V\alpha}{r^2}\right)\eta = 0 \tag{1}$$

with

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}, \tag{2}$$

$$\alpha = (3\gamma - 4)/\gamma, \tag{3}$$

$\eta = \delta r/r$, $V = -d \ln P/d \ln r$, and $\sigma = 2\pi/\Pi$, where Π is the period. It is assumed that γ, the ratio of specific heats of the gas, is a constant.

Pulsational eigenfrequencies for the five models with known analytic solutions follow.

Homogeneous model (Sterne 1937):

$$\sigma_j^2 = GMR^{-3}\gamma[j(2j+5)+\alpha] \qquad j = 0, 1, 2, \ldots. \tag{4}$$

Inverse-square model (Sterne 1937):

$$\sigma_j^2 = GMR^{-3}(\gamma/2)(2j+q)(2j+q+3) \qquad j = 0, 1, 2, \ldots \tag{5a}$$

with

$$q = \tfrac{1}{2}[(1+8\alpha)^{1/2} - 1]. \tag{5b}$$

Roche model (Sterne 1937):

$$\sigma_j^2 = GMR^{-3}(\gamma/3)(3j+q)(3j+q+3) \qquad j = 0, 1, 2, \ldots \tag{6a}$$

with

$$q = (3\alpha)^{1/2}. \tag{6b}$$

Prasad model (Prasad 1948):

$$\sigma_j^2 = GMR^{-3}(\gamma/3)[(3j+q)(3j+q+5)+2\alpha] \qquad j = 0, 1, 2, \ldots \tag{7a}$$

with

$$q = (1+\alpha)^{1/2} - 1. \tag{7b}$$

The four models so far listed may be conveniently arranged in order of their interior density distributions, which might, on general grounds, be expected to govern their pulsational characteristics. Since $\rho_c/\langle\rho\rangle$ is infinite for the last three models discussed, a better indicator of the models' central condensation is $\langle\rho\rangle/\rho_R$ or, as Singh (1968) pointed out in the case of composite polytropes, the total moment of inertia about the centre, $I = \int r^2\, dM(r)$.

Table 1. Pulsational characteristics as a function of central condensation.

Model	$\rho_c/\langle\rho\rangle$	$\langle\rho\rangle/\rho_R$	I/MR^2	ω_0^2 ($\gamma = 5/3$)	Π_1/Π_0 ($\gamma = 5/3$)	Π_2/Π_0 ($\gamma = 5/3$)
Homogeneous	1	1	3/5	1.00	0.281	0.180
Prasad	∞	3/2	2/5	1.44	0.303	0.190
Inverse square	∞	3	1/3	2.17	0.411	0.268
Roche	∞	∞	0	3.24	0.427	0.277

The pulsational characteristics for $\gamma = 5/3$ are given in Table 1, where $\omega_j^2 = \sigma_j^2\, R^3/GM$. It is a strange mathematical accident that the general relation between ω_0^2 and γ can be represented as $\gamma = (4 + \omega_0^2)^2/(12 + k\omega_0^2)$ with $k = 3$, 4, 5, and 6 for the homogeneous, Prasad, inverse-square, and Roche models, respectively.

Atmospheric model (Lamb 1932; Gough *et al.* 1965): the pulsational eigenfrequencies are given by

$$\sigma_j^2 = GMR^{-3}(\gamma/4)j_n^2(n+1)^{-1}(1-x_c)^{-1} \qquad j = 0, 1, 2, \ldots \tag{8}$$

where n is the polytropic index of the atmosphere, x_c is the radius fraction of the atmosphere's inner boundary, and j_n represents a zero of the Bessel function J_n. Equation (8) follows as a solution of equation (1) if use is made of the fact that, in an atmosphere, $1 \ll V \ll (\sigma^2 \rho R^2/\alpha\gamma P)$. This model and its variants are discussed in an astrophysical context by Ledoux & Walraven (1958) as well as by Gough *et al.* (1965).

2.2 SOLUTIONS INCLUDING AXIAL ROTATION

Axial rotation will be included in the models by making three simplifying assumptions, namely, (1) that spherical symmetry is formally preserved; (2) that the oscillations remain purely radial; and (3) that each mass shell conserves its angular momentum during the oscillations. The necessary dynamical equations, in which all forces are averaged over a spherical shell, have been derived elsewhere (Stothers 1974). They are here combined into one equation analogous to the Eddington equation:

$$\frac{d^2\eta}{dr^2} + \left(\frac{4-V}{r}\right)\frac{d\eta}{dr} + \left[\frac{\sigma^2\rho}{\gamma P} - \frac{V}{r^2}\left(\alpha + \frac{1}{\gamma}\frac{\lambda}{1-\lambda}\right)\right]\eta = 0 \tag{9}$$

with

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}(1-\lambda), \tag{10}$$

where $\lambda = (2/3)\Omega^2 r^3/GM(r)$ and Ω is the angular velocity of rotation.

Notice that if λ is a constant (i.e. the mean ratio of centrifugal force to gravity is the same at every layer), then equation (9) has the same form as equation (1). It follows that the corresponding eigenfrequencies of pulsation for the five stellar models discussed above are still represented by equations (4)–(8), but with the following substitutions:

$$\alpha \rightarrow \alpha + (\lambda/\gamma)(1-\lambda)^{-1}, \tag{11}$$

$$G \rightarrow G(1-\lambda). \tag{12}$$

The influence of rotation on the models is thus manifested both through the alteration of the equilibrium structure (via the change in G) and through the direct interaction with the oscillations (via the change in α). It has not been possible to find other distributions of λ that would admit solutions in closed form.

In the case of the homogeneous model, a constant value of λ implies a constant value of Ω. Simon (1969, 1970) has previously considered the uniformly rotating homogeneous model, and has included the rotational distortion of the star. It is interesting to find that the pulsational eigenfrequencies in these two different studies agree exactly. Ledoux's (1945) and Cowling & Newing's (1949) even earlier studies provided an approximate expression for the fundamental pulsational eigenfrequency of any slowly rotating star; in the case of the homogeneous model, their approximate expression becomes exact and equal to ours.

For the more centrally condensed models, our results are entirely new. A curious feature of ω_0^2 arises in all the stellar models: ω_0^2 is found to be an increasing function of λ for $\gamma < \gamma_c$ and a decreasing function of λ for $\gamma > \gamma_c$. Since the function generally exhibits some curvature, our remark strictly applies only at small λ. The explanation of this strange feature is that the direct interaction of rotation with the oscillations always increases ω_0^2, while the change in equilibrium structure of the star always lowers it; when $\gamma = \gamma_c$, the two opposing effects balance. We find that, in general, a quadratic equation exists for γ_c, whose relevant root is $\gamma_c = 5/3$, 1.6561, 1.6153 and 1.5450 for the homogeneous, Prasad, inverse-square and Roche models, respectively. A similar feature has been reported for models of uniformly rotating polytropes (Chandrasekhar & Lebovitz 1968; see also Clement 1965), but our explanation of the feature, being based on easily separable analytic factors, seems more rigorous. In all cases, γ_c decreases with increasing central condensation of the star, and obeys nearly the same functional dependence on ω_0^2.

Fig. 1 shows the quantities ω_0^2, Π_1/Π_0, and Π_2/Π_0 for the Roche model compared with those for the homogeneous model. Unless γ is close to 4/3, the plotted quantities display surprisingly little sensitivity to the assumed rate of rotation. Based as they are on a wide range of angular momenta, these results for the Roche model (which should be representative of real, centrally condensed stars) can readily explain the similar near constancy of pulsation period derived for detailed models of rotating classical Cepheids (Carson & Stothers 1976; Cox *et al.* 1977; Deupree 1978). The new results also lead to a prediction that rotation should markedly decrease the fundamental pulsation period of rotating variable stars with γ close to 4/3 (e.g. red giants and massive white dwarfs).

2.3 SOLUTIONS INCLUDING MAGNETIC FIELDS

In analogy with the preceding treatment of rotation, magnetic fields will be introduced into the models under the simplifying assumptions that spherical symmetry is preserved, that the oscillations are radial, and that magnetic flux is conserved locally during the course of the oscillations. By using dynamical equations derived for the axisymmetric case where $\langle H_r^2 \rangle = (1/3)\ \langle H^2 \rangle$ (Stothers 1979a) and by assuming that the resulting pseudo-isotropic pressure of the magnetic field, $\langle H^2 \rangle / 24\pi$, is proportional to the local gas pressure raised to the ath power, it is readily shown that Eddington's equation must be modified to read

$$\frac{d^2\eta}{dr^2} + \left[\frac{4}{r} - \frac{V}{r}\left(\frac{1 + 4a\nu/3\gamma}{1 + 4\nu/3\gamma}\right)\right]\frac{d\eta}{dr} + \left(\frac{1}{1 + 4\nu/3\gamma}\right)\left(\frac{\sigma^2\rho}{\gamma P} - \frac{V\alpha}{r^2}\right)\eta = 0 \tag{13}$$

with

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}\left(\frac{1}{1 + a\nu}\right) \tag{14}$$

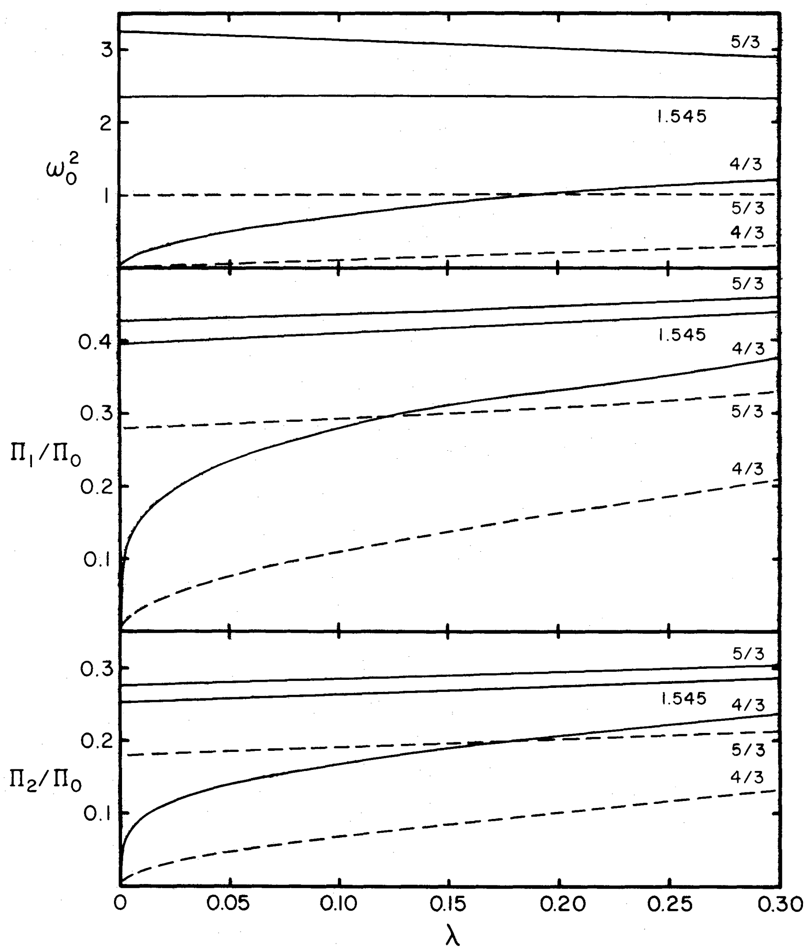

Figure 1. Pulsational quantities for three values of γ in rotating star models. The Roche model is indicated by solid lines, the homogeneous model by dashed lines.

and $\nu = \langle H^2 \rangle / 24\pi P$. At the surface of the stellar model, the magnetic field is assumed to vanish.

If ν is taken to be constant (and hence the parameter a is unity), equation (13) has the same form as equation (1). Consequently, the five standard analytic solutions apply also to the present case, provided that the following replacements are made

$$\alpha \to \alpha(1 + 4\nu/3\gamma)^{-1}, \tag{15}$$

$$G \to G(1 + \nu)^{-1}, \tag{16}$$

$$\sigma^2 \to \sigma^2(1 + 4\nu/3\gamma)^{-1}. \tag{17}$$

The change in G simply represents the alteration of the equilibrium structure of the model, whereas the changes in α and σ^2 reflect magnetic interactions with the oscillations. Note that the oscillations are not Alfvénic in character, but are gravitationally controlled.

The task of obtaining analytic solutions for models with nonconstant ν seems to be much harder, except in one trivial case. If the mean magnetic field $\langle H^2 \rangle^{1/2}$ is constant and does not vanish at the stellar surface, the pulsational eigenfrequencies for the homogeneous model turn out to be the same as in the non-magnetic case.

Some earlier analytic work can now be compared with our present results. For the homogeneous model with constant ν, Chandrasekhar & Limber's (1954) approximate virial

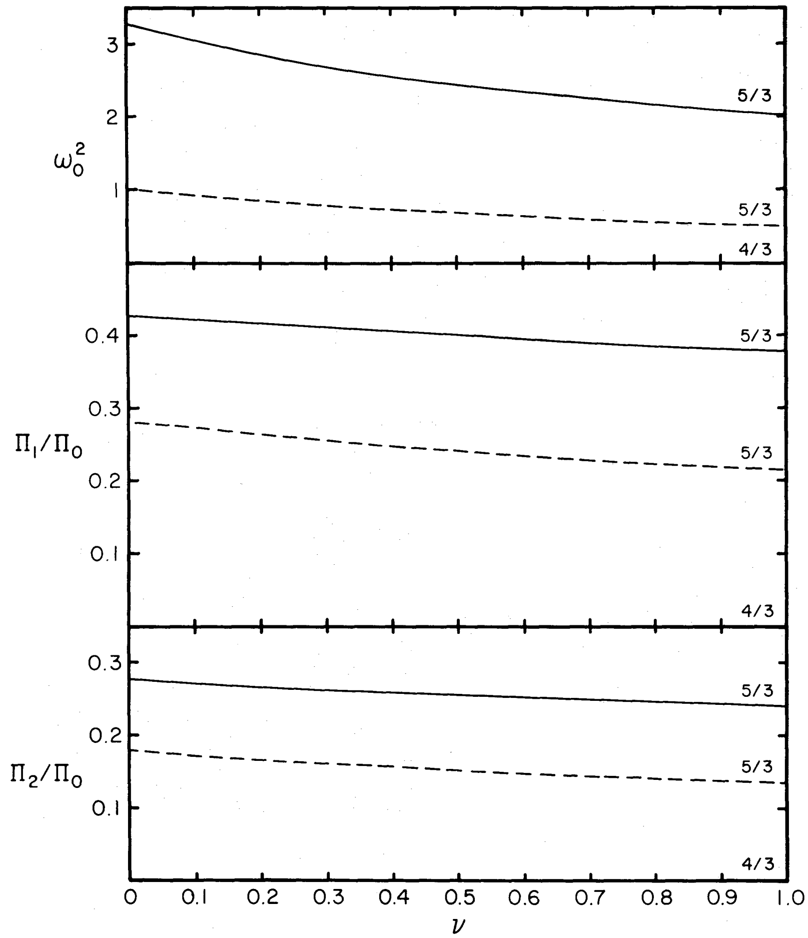

Figure 2. Pulsational quantities for two values of γ in magnetic star models. The Roche model is indicated by solid lines, the homogeneous model by dashed lines.

expression for the fundamental pulsational eigenfrequency of a magnetic star reduces to an exact expression, and equals our result. In the case of a purely uniform magnetic field, Ledoux & Simon's (1957) approximate solution, derived by a perturbation method and under the assumption that the magnetic field does not vanish at the stellar surface, is the same as our related solution for the case of constant $\langle H^2 \rangle^{1/2}$. Those authors have considered, in a similar way, the inverse-square model, and have obtained a result that bears a qualitative resemblance to the one we have obtained on the basis of constant ν.

Among the various analytic models with constant ν, the Roche model is the most representative of real stars, because most stars have a very high central condensation. Fig. 2 shows for the Roche model, in comparison with the homogeneous model, the quantities ω_0^2, Π_1/Π_0, and Π_2/Π_0. In both cases these quantities decrease with increasing ν, except when $\gamma = 4/3$, in which case they are all zero. Evidently the change in equilibrium structure, which reduces ω_0^2, always outweighs the direct magnetic interaction with the oscillations, which tends to increase ω_0^2. A similar decrease of ω_0^2 has been obtained for models of polytropes containing mixed poloidal and toroidal magnetic fields (e.g. Trehan & Uberoi 1972; Sood & Trehan 1975). But the present Roche model confirms in a more straightforward way the published models of classical Cepheids possessing tangled magnetic fields (Stothers 1979b). Other variable stars with complicated magnetic structures (e.g. stars containing convective envelopes) can now also be expected to show the same qualitative dependence of their pulsational periods on ν.

3 Stability

The one-zone model of a star (Jeans 1927, 1929; Baker 1966) can be used to discuss the problem of stability in the linear approximation. More sophisticated approaches exist (e.g. Ledoux 1963), but little would be gained by using them here. Dynamical stability has already been treated rigorously, since the criterion for its existence is the same as the condition that the fundamental pulsational eigenfrequency exceed zero. Pulsational stability depends, in an accurate treatment, on the radial distribution of the amplitudes, which are of course here available. But as long as one is going to resort to a distributed model of some sort, the radial distribution of the thermodynamical and atomic coefficients should also be included (e.g. Cox 1974), which would lead immediately to a problem of numerical quadratures. Secular stability requires, in a precise investigation, generally unavailable information about the detailed form of the perturbation to the star. Under the simplest assumption of a homologous perturbation, the one-zone model itself gives an exact result (Jeans 1927, 1929; Ledoux 1963). Generally speaking, this simple model has proven to be the most successful 'average' model of a variable star (Okamoto & Unno 1967).

In the present approach, it will be both convenient and consistent to set the spatial derivatives of perturbed quantities equal to zero, except of course for the spatial derivative of the luminosity perturbation, which is the source of the required non-adiabatic effects. Baker's (1966) approximation for this derivative is

$$\frac{d}{dM(r)}\left(\frac{\delta L}{L}\right)=\frac{2}{\Delta M}\left(\frac{\delta L}{L}\right), \tag{18}$$

where δL is the mean value of the luminosity perturbation in the zone being considered and ΔM is the mass of the zone. Neglecting nuclear energy sources and convection, and introducing axial rotation and magnetic fields in the approximations of Section 2, we find for the composite equation of motion

$$\frac{\partial^3\eta}{\partial t^3}+KwA\,\frac{\partial^2\eta}{\partial t^2}+w^2B\,\frac{\partial\eta}{\partial t}+Kw^3D\eta=0, \tag{19}$$

where

$$A=\frac{\kappa_{\mathrm{P}}\rho_{\mathrm{T}}-(\kappa_{\mathrm{T}}-4)\rho_{\mathrm{P}}}{\rho_{\mathrm{T}}+\rho_{\mathrm{P}}C};$$

$$B=\frac{3YC-X(\rho_{\mathrm{T}}+\rho_{\mathrm{P}}C)-3Z}{\rho_{\mathrm{T}}+\rho_{\mathrm{P}}C},$$

$$D=\frac{(\kappa_{\mathrm{T}}-4)\,(X\rho_{\mathrm{P}}-3Y)-\kappa_{\mathrm{P}}(X\rho_{\mathrm{T}}+3Z)-4(Y\rho_{\mathrm{T}}+Z\rho_{\mathrm{P}})}{\rho_{\mathrm{T}}+\rho_{\mathrm{P}}C},$$

$$K=-\frac{2L\rho}{P\rho_{\mathrm{T}}w\Delta M};\qquad C=\frac{\Gamma_2}{\Gamma_2-1};\qquad w^2=\frac{GM(r)}{r^3}; \tag{20}$$

$$X=4-5\lambda;\qquad Y=1-\lambda+(\tfrac{4}{3}\rho_{\mathrm{P}}-1)\mu;\qquad Z=-\tfrac{4}{3}\rho_{\mathrm{T}}\mu;$$

$$\rho_{\mathrm{P}}=\left.\frac{\partial\ln\rho}{\partial\ln P}\right|_{\mathrm{T}};\qquad \rho_{\mathrm{T}}=\left.\frac{\partial\ln\rho}{\partial\ln T}\right|_{\mathrm{P}};\qquad \kappa_{\mathrm{P}}=\left.\frac{\partial\ln\kappa}{\partial\ln P}\right|_{\mathrm{T}};\qquad \kappa_{\mathrm{T}}=\left.\frac{\partial\ln\kappa}{\partial\ln T}\right|_{\mathrm{P}};$$

$$\lambda=\frac{2}{3}\,\frac{\Omega^2r^3}{GM(r)};\qquad \mu=-\frac{4\pi r^4}{GM(r)}\,\frac{d}{dM(r)}\left(\frac{\langle H^2\rangle}{24\pi}\right).$$

Assuming a time dependence of η in the form $\exp(st)$, we have from equation (19)

$$s^3 + KwAs^2 + w^2Bs + Kw^3D = 0. \tag{21}$$

As Jeans originally showed, the criterion that all three roots have negative real parts, i.e. that the star be stable, can be written

$$w^2B > 0 \quad \text{(dynamical stability)} \tag{22}$$

$$Kw^3D > 0 \quad \text{(secular stability)} \tag{23}$$

$$Kw^3(AB - D) > 0 \quad \text{(pulsational stability).} \tag{24}$$

Notice that w is a positive quantity. We shall assume that K too is positive, as it will be for an ideal gas. Also, for an ideal gas, $\Gamma_2 = \gamma$, $\rho_P = 1$, and $\rho_T = -1$. Finally, in terms of the parameter $\nu = \langle H^2 \rangle / 24\pi P$, we have $\mu = \nu/(1+\nu)$ if ν is constant in space.

3.1 ROTATIONAL CASE

Stability in this case exists under the conditions

$$3\gamma - 4 + (5 - 3\gamma)\lambda > 0 \quad \text{(dynamical)} \tag{25}$$

$$(\kappa_T - 4)(1 - 2\lambda) + \kappa_P(4 - 5\lambda) + 4(1 - \lambda) > 0 \quad \text{(secular)} \tag{26}$$

$$(\kappa_T - 4)(1 - \gamma) - \kappa_P\gamma - \tfrac{4}{3} > 0 \quad \text{(pulsational).} \tag{27}$$

Generally, rotation acts to stabilize the star dynamically. This result has been known for a long time (e.g. Ledoux 1945). The condition for secular stability in the case of Kramers's opacity ($\kappa_P = 1$, $\kappa_T = -9/2$) is $\lambda > 1/16$, or in the case of electron-scattering opacity ($\kappa_P = 0$, $\kappa_T = 0$), $\lambda > 0$. Even if nuclear energy sources are entirely absent a small amount of rotational angular momentum can prohibit the slow gravitational contraction of a star. Apparently, this interesting possibility has not yet emerged from studies of the evolution of rotating protostars. Pulsational stability, on the other hand, turns out to be completely unaffected by rotation. This simple result explains why the blue edge of the theoretical instability strip for classical Cepheids has been found to be essentially independent of the assumed rate of rotation (Carson & Stothers 1976). It further predicts that other classes of pulsating variable stars should, in this respect, be insensitive to rotation.

3.2 MAGNETIC CASE

The stability conditions are now

$$3\gamma - 4 > 0 \quad \text{(dynamical)} \tag{28}$$

$$\kappa_T + 4\kappa_P > 0 \quad \text{(secular)} \tag{29}$$

$$(\kappa_T - 4)(1 - \gamma) - \kappa_P\gamma - \tfrac{4}{3} > 0 \quad \text{(pulsational).} \tag{30}$$

It is a remarkable fact that stellar magnetism (of the adopted type) has no influence on the stability of a star. This appears to be a consequence as much of the assumption that magnetic flux is conserved during the displacements as of the assumption that $\langle H_r^2 \rangle = (1/3)\ \langle H^2 \rangle$; these two assumptions cause the magnetic field lines to behave like a gas with $\gamma = 4/3$, i.e.

the displacements become homologous. Consequently, our results may be somewhat more general than at first appears. However, we shall compare them here only with published models based on similar assumptions. The only relevant models in this case refer to classical Cepheids, whose pulsational instability has been found to be practically independent of the assumed strength of the magnetic field (Stothers 1979b). It follows that similar results can be expected for the pulsational instability of related classes of variable stars.

4 Conclusions

Eddington's form of the wave equation for small-amplitude, radial, adiabatic stellar pulsations has been generalized to include, in a simple way, the effects of axial rotation and of tangled magnetic fields. Solutions in closed form are obtainable for four analytic stellar models (and a stellar atmosphere model) if the ratio of mean centrifugal force to gravity, λ, and the ratio of mean magnetic pressure to gas pressure, ν, are constant throughout the star. Under the present assumptions, the central condensations of the models (all the models are well known from earlier work) are unchanged by the presence of the rotational and magnetic forces. However, equilibrium quantities possessing dimensions are of course affected in general. We find that rotation produces exactly the same structural modifications in the models as does a magnetic field if $(1-\lambda)=(1+\nu)^{-1}$. However, the relative importance of rotation and magnetism in affecting the pulsational characteristics of the models depends sensitively on the choices of γ and of the type of model.

Nevertheless, certain general conclusions can be drawn. If we consider the range $4/3 \leqslant \gamma \leqslant 5/3$ and if we make an exception of the fundamental eigenfrequency of the rotating star models with $\gamma \geqslant \gamma_c$, we find that rotation generally increases ω_0^2, Π_1/Π_0, and Π_2/Π_0, and that magnetism generally decreases these quantities. Pulsationally, rotation and magnetism lead to the same kind of effects as those that accompany, respectively, an increase and a decrease of central condensation (see Table 1). Again with the exception mentioned above, and away from the neighbourhood of $\gamma=4/3$, both rotation and magnetism have a proportionately larger influence on the pulsational characteristics of the models if the models possess lower central condensations. In this connection, a lower central condensation leads also to a larger value of γ_c. Finally, if either $\gamma=4/3$ or $\nu=\infty$, the fundamental eigenfrequency of the non-rotating models becomes zero (as has long been known). These completely analytic results accord with, and partly extend, the detailed numerical results derived previously for certain types of rotating and magnetic polytropes, as well as for actual models of classical Cepheids. They also lead to obviously similar predictions for related classes of variable stars.

Stability has been considered in the case of the homogeneous stellar model by a generalization of the one-zone model of Jeans and Baker. Rotation is found to provide dynamical and secular stability, but to have no effect on the pulsational stability. Magnetism turns out not to affect any of the stability criteria. These results clarify the detailed results already derived for classical Cepheid models, and lead to predictions concerning the stability of rapidly rotating protostars and variable red giants. Since all our results have sprung from rather simply understood stellar models covering a large range of central condensations, rotational angular momenta, and magnetic energies, they are probably quite widely applicable to more realistic cases in which our assumptions about the rotational and magnetic fields remain approximately valid. However, it must be remembered that our results refer only to radial perturbations and that more complicated displacements could lead to very different results.

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